8D

SOME EXPLICIT ESTIMATES AND APPLICATIONS IN PRIME NUMBER THEORY

by

MARILYNN L. FAULKNER

A THESIS

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES

IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE

OF DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS

UNIVERSITY OF ALBERTA

EDMONTON, ALBERTA

SEPTEMBER, 1966

UNIVERSITY OF ALBERTA

FACULTY OF GRADUATE STUDIES

The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies for acceptance, a thesis entitled "SOME EXPLICIT ESTIMATES AND APPLICATIONS IN PRIME NUMBER THEORY", submitted by MARILYNN L. FAULKNER in partial fulfilment of the requirements for the degree of Doctor of Philosophy.

ABSTRACT

The statement "The product of k consecutive integers, each greater than k, has a prime divisor greater than k" is commonly known as the Sylvester-Schur theorem. In the first chapter we prove a refinement of this theorem which is, in a sense, best possible: If $n \geq k$ then $(n+1)(n+2)\ldots(n+k)$ has a prime divisor greater than or equal to $\frac{7}{5}k$.

The question, posed by L. Moser, of the existence of an analogue of the Sylvester-Schur theorem for primes in arithmetic progression led to the problem of estimating certain functions of primes in arithmetic progression. A method is developed in the second chapter for estimating

$$\psi(x;a,b) = \sum_{\substack{p^\alpha \equiv b(a) \\ p^\alpha \leq x}} \log p$$

where $(a,b) = 1$, and it is proved there that $.40x < \psi(x;4,b) < .59x$ for $b = 1, 3$, $x \geq 43$.

The next chapter contains two completely elementary and different proofs of the known estimate

$$\sum_{\substack{p \equiv 1(4) \\ p \leq x}} \frac{\log p}{p} = \frac{1}{2} \log x + O(1) \quad .$$

With only slight modifications these methods apply also to the case $p \equiv 1(3)$.

In studying the solvability of the diophantine equation $n! = x^4 - y^4$, for $(x,y) = 1$, relatively sharp estimates for $\psi(x;4,1)$, $\psi(x;4,3)$ and $\sum_{\substack{p \equiv 1(4) \\ p \leq x}} \frac{\log p}{p}$ are required. Thus, with the results of Chapter II and the methods of Chapter III, we prove in Chapter IV that the equation $n! = x^4 - y^4$ has no solution in positive integers n,x,y with $(x,y) = 1$, a result already known for n sufficiently large.

ACKNOWLEDGEMENTS

I would like to express my gratitude to Professor Leo Moser for the inspiration and encouragement he has provided throughout the supervision of this thesis and for the kindness he has extended to me during my graduate career.

I would also like to thank the University of Alberta and the National Research Council for the financial support they have given me.

TABLE OF CONTENTS

CHAPTER I

ON A THEOREM OF SYLVESTER AND SCHUR

The theorem mentioned in the title, which was first proved by J. Sylvester [28] in 1892 and later rediscovered and proved by I. Schur [26] in 1929, asserts that the product of k consecutive integers, each greater than k, has a prime divisor greater than k. In this chapter we will prove a recent conjecture of P. Erdős and obtain as a corollary a "best-possible" refinement of the Sylvester-Schur theorem announced by L. Moser [19].

Theorem. Let p_k be the least prime $\geq 2k$. If $n \geq p_k$ then $\binom{n}{k}$ has a prime divisor $\geq p_k$, with the exceptions $\binom{9}{2}$, $\binom{10}{3}$.

The example $6\cdot 7\cdot 8\cdot 9\cdot 10$ shows that the following refinement of the Sylvester-Schur theorem is best possible:

Corollary. If $n \geq 2k$ then $\binom{n}{k}$ has a prime divisor $\geq \frac{7}{5}k$.

Throughout the proof of the theorem and corollary, p denotes a prime and $p^\alpha \| m$ indicates that $p^\alpha | m$ but $p^{\alpha+1} \nmid m$. Recently J. Rosser and L. Schoenfeld [22] obtained the following fairly precise estimates for $\theta(x) = \sum_{p\leq x} \log p$ and $\pi(x) = \sum_{p\leq x} 1$:

$$(1.1) \qquad \theta(x) < 1.01624x \qquad \text{for} \quad x > 0\ ,$$

$$(1.2) \qquad \pi(x) < 1.25506\,\frac{x}{\log x} \qquad \text{for} \quad x > 1\ .$$

Using these results we are able to establish the theorem by methods which are similar to those used by P. Erdős [7] in his elegant proof of the Sylvester-Schur theorem.

Proof of the theorem. Assume $\binom{n}{k}$ has no prime divisors $\geq 2k$. Since $p^{\alpha} \mid \binom{n}{k}$ implies $p^{\alpha} \leq n$, (see the paper of P. Erdős [7]),

$$\left(\frac{n}{k}\right)^{k} \leq \binom{n}{k} = \prod_{\substack{p^{\alpha} \| \binom{n}{k} \\ p < 2k}} p^{\alpha} \leq n^{\pi(2k)}\ .$$

From (1.2) it follows that

$$\begin{aligned} \frac{n}{k} &< n^{2(1.25506)/\log 2k} \\ &< n^{1/2} \qquad \text{for} \quad k \geq 76\ . \end{aligned}$$

Thus the assumption is false if $76 \leq k \leq n^{1/2}$.

Let $k > n^{1/2}$, then by (1.1) and (1.2)

$$\left(\frac{n}{k}\right)^k \le \binom{n}{k} \le \prod_{p<2k} p \cdot \prod_{\substack{p^\alpha \| \binom{n}{k} \\ p \le \sqrt{n}}} p^\alpha$$

$$< e^{1.01624(2k)} \cdot n^{2(1.25506)\sqrt{n}/\log n} \quad . \tag{1.3}$$

Hence

$$\left(\frac{n}{k}\right)^k < e^{1.01624(2k) + 2(1.25506)\sqrt{n}} \quad . \tag{1.4}$$

Replacing $\sqrt{n}$ by k and taking k^{th} roots in (1.4) yields $n < e^{4.55}\,k$. However, when $k > e^{-4.55}\,n$ we may replace $\sqrt{n}$ by $e^{4.55}\frac{k}{\sqrt{n}}$ in (1.4) and take k^{th} roots to obtain

$$n < e^{2(1.01624) + 2(1.25506)e^{4.55}/\sqrt{n}} \cdot k$$

$$< e^{3.49}\,k$$

provided $n > 3 \cdot e^{9.1}$, that is for $n > 26{,}900$. The initial assumption is then false if $n^{1/2} < k \le e^{-4.55}\,n$ or $e^{-4.55}\,n < k \le e^{-3.49}\,n$ with $n > 26{,}900$.

A simple induction argument shows that $\binom{9k}{k} > \frac{(21.3)^k}{3k}$ for $k = 1,2,3,\ldots$. Thus, for $9k \le n < e^{3.49}\,k$, it follows from (1.3) that

$$\frac{(21.3)^k}{3k} \le \binom{n}{k} < e^{1.01624(2k) + 2(1.25506)\sqrt{n}} \quad .$$

Taking k^{th} roots we obtain

$$(1.5) \qquad 21.3 < e^{2(1.01624) + 2(1.25506)\frac{\sqrt{n}}{k} + \frac{\log 3k}{k}} .$$

Since $\frac{\sqrt{n}}{k} < \frac{e^{3.49}}{\sqrt{n}} < \frac{e^{3.49}}{\sqrt{26,900}}$ and $\frac{\log 3k}{k} \leq \frac{\log 228}{76}$ for $26,900 < n < e^{3.49} k$ and $k \geq 76$, a simple calculation shows that inequality (1.5) is false. Our initial assumption is therefore false for $9k \leq n < e^{3.49} k$ with $n > 26,900$ and $k \geq 76$.

R. Breusch [3] proved the existence of a prime in the interval $(m, \frac{9}{8} m)$ for $m \geq 48$. Hence there is a prime $p \geq p_k$ such that $p | \binom{n}{k}$ for $n \geq 54$ and $3k \leq n < 9k$.

Clearly $p_k | \binom{n}{k}$ if $2k \leq p_k \leq n < 3k$. Thus the theorem is proved with the exception of a number of special cases which can be separated into the two categories: (i) $k < 76$ with $n \geq 3k$ and (ii) $k \geq 76$ with $9k \leq n \leq 26,900$. Case (ii) follows from the fact that the maximum gap between consecutive primes less than 26,900 is less than 76 as shown by the table compiled by K. Appel and J. Rosser [1] or by the table included in the paper of J. Sylvester [28]. To settle case (i), various special arguments are required.

For $k = 1$ the theorem is obviously true, while for $k = 2,3$ an argument like that used by W. Utz [30] is needed. W. Utz proved that if $f(k)$ denotes the least integer such that each product of $f(k)$ consecutive integers, all greater than k , has a prime factor greater than k , then $f(7) = f(8) = f(9) = f(10) = 4$. He established this

result by means of a proof by contradiction in which the resulting diophantine equations were shown to be insolvable. Thus, for $k = 2$ and $n \neq 9$ the theorem follows from the insolvability of $2^r \pm 1 = 3^s$ in positive integers r, s with $r > 3$. A similar argument shows that the theorem holds for $k = 3$ with the exception of $\binom{10}{3}$. The case $k = 4$ follows directly from the result of W. Utz [30].

D. Lehmer [16] has shown recently that every set of 7 consecutive integers greater than 36 contains a multiple of a prime greater than or equal to 43. Therefore, since the maximum gap between successive primes less than 57 is 6, the theorem holds without exception for $7 \leq k \leq 21$.

Now, if the theorem is violated for some n with $k = 5$, or 6, or $21 < k < 76$ then n must satisfy the inequality

$$(1.6) \qquad \frac{n^k}{k!}\left(1 - \frac{k(k+1)}{2n}\right) \leq \frac{n^k}{k!}(1-\frac{1}{n})(1-\frac{2}{n})\ldots(1-\frac{k-1}{n}) = \binom{n}{k} \leq n^{\pi(2k)} .$$

Thus, if $k = 5$, it follows from (1.6) that $\frac{n^5}{5!}(1-\frac{15}{n}) \leq n^4$, which is false for $n > 135$. Similarly, if $k = 6$, inequality (1.6) is false for $n > 741$. Direct checking for $n \leq 135$ with $k = 5$ and $n \leq 741$ with $k = 6$ completes the proof in these cases. For $n \geq \frac{(k+1)^2}{2}$, inequality (1.6) implies

$$(1.7) \qquad n^{k-\pi(2k)} \leq (k+1)!$$

and, as computations reveal, $(k+1)!^{1/(k-\pi(2k))} < 1000$ for $21 < k < 76$. Thus inequality (1.7) is false for $21 < k \leq 44$ with $n \geq 1000 > \frac{(k+1)^2}{2}$ and for $44 < k < 76$ with $n \geq 2850 > \frac{(k+1)^2}{2}$. To complete the proof of the theorem, we refer to the Appel-Rosser table [1] or Sylvester's table [28] to find that the maximum gap between successive primes less than 1000 is less than 21, while for primes less than 2850 the maximum gap is less than 44.

Proof of the corollary. By the theorem just proved, $\binom{n}{k}$ has a prime factor $\geq \frac{7}{5} k$ if $n \geq p_k$. The result of R. Breusch [3] referred to earlier, shows that both the intervals $(\frac{7}{5} k, 2k)$ and $(2k, \frac{9}{4} k)$ contain primes for $k \geq 35$. Thus, for $n < p_k < \frac{9}{4} k$, the interval $(n-k, n)$ contains $(\frac{7}{5} k, 2k)$ which contains a prime. The proof is completed by verifying the corollary directly for $k < 35$ with $2k \leq n < p_k$.

The above theorem and corollary, together with their proofs, have been published in the Journal of the London Mathematical Society [12].

M. Subba Rao, in conversation with the author, suggested the following refinement of the Sylvester-Schur theorem: For $n \geq k \geq k_0$, the product $(n+1)(n+2)\ldots(n+k)$ has two prime divisors greater than k. The existence of such a k_0 follows from the results of P. Erdős [9] who proved that if $g(k)$ is the smallest integer so that among k consecutive integers, each greater than k, there are at least $g(k)$ of them having prime factors greater than k, then $g(k) = (1 + o(1))\frac{k}{\log k}$. By the methods used in the proof of the

theorem at the beginning of this chapter, it follows easily that $k_o \leq 14$, and a variety of special arguments similar to those given in the proof could probably reduce this number to 3.

In connection with the Sylvester-Schur theorem, L. Moser has posed the following question:

(1.8) Let a,b be positive integers with $(a,b) = 1$, $1 \leq b \leq a-1$, $a > 2$. Further, let $n \geq k \geq \phi(a)$, where ϕ denotes Euler's totient function. Then, is it true that the product $(n+1)(n+2)\ldots(n+k)$ has a prime divisor $p > k$ with $p \equiv b(\text{mod } a)$?

The author feels that the answer to this question is very likely negative, for otherwise, as we will show, it would contradict Hypothesis H of A. Schinzel [24;25]:

(1.9) Let $f_1(x), f_2(x)\ldots, f_n(x)$ be n irreducible polynomials with integer coefficients, where the coefficient of the highest power of x in every case is positive. If these polynomials satisfy the condition that there exists no integer > 1 which divides the product $f_1(x)f_2(x)\ldots f_n(x)$ for all integers x then there are infinitely many integers x for which $f_1(x), f_2(x), \ldots, f_n(x)$ are simultaneously prime.

While this hypothesis has many interesting consequences, the only special case for which it is known to hold is for $n = 1$ - the hypothesis reducing then to Dirichlet's theorem. P. Bateman and R. Horn [2] have given

Hypothesis H a quantitative interpretation and, in addition, have done some numerical investigation to illustrate the plausibility of their formulae in a few very special cases.

Now, with a and b as in (1.8), let h_k denote the least common multiple of the integers $1,2,3,\ldots,k$ and set $n_m = \frac{h_k}{m}$, $m = 1,2,\ldots,k$. Consider

$$P_1(x) = (ah_kx+1)(ah_kx+2)\ldots(ah_kx+k)$$

$$= k!\ (an_1x+1)(an_2x+1)\ldots(an_kx+1)$$

and

$$P_2(x) = (ah_kx-1)(ah_kx-2)\ldots(ah_kx-k)$$

$$= k!\ (an_1x-1)(an_2x-1)\ldots(an_kx-1) \quad .$$

Then clearly the two sets of polynomials

$$\{an_1x+1, an_2x+1, \ldots, an_kx+1\}\ , \quad \{an_1x-1, an_2x-1, \ldots, an_kx-1\}$$

each satisfy the conditions of (1.9). It therefore follows from (1.9) that there exists an infinite set S_1 of integers x for which $(an_1x+1),\ldots,(an_kx+1)$ are simultaneously prime for $x \in S_1$, and, similarly, an infinite set S_2 of integers x for which $(an_1x-1),\ldots,(an_kx-1)$ are simultaneously prime. Thus $P_1(x)$ is a product of k consecutive integers which has no prime divisor

$p > k$, $p \equiv b(\mathrm{mod}\ a)$, $b \nmid 1$ for each $x \in S_1$; $P_2(x)$ is a product of k consecutive integers which has no prime divisor $p > k$, $p \equiv b(\mathrm{mod}\ a)$, $b \nmid -1$ for each $x \in S_2$.

CHAPTER II

ON THE ESTIMATION OF THE ψ-FUNCTION FOR PRIMES IN ARITHMETIC PROGRESSION

It is the aim of this chapter to develop a method of estimating the function

$$\psi(x;a,b) = \sum_{\substack{p^\alpha \equiv b(a) \\ p^\alpha \leq x}} \log p \tag{2.1}$$

for particular a and b, where $(a,b) = 1$, $a > 2$, p prime, and we will use this method to prove that

$$.40x < \psi(x;4,b) < .59x \tag{2.2}$$

for $b = 1,3$, $x \geq 43$. One of the motivations for studying this function was the attempt to obtain an analogue of the Sylvester-Schur theorem for primes in arithmetic progression.

First of all, let us describe the results found in the literature which are related to the above problem. These results are concerned with either an attempt to supply an elementary proof of the classical Dirichlet theorem or an attempt to solve the following problem for particular a,b and $\epsilon > o$: Determine $n_o = n(\epsilon,a,b)$ so that the interval $(n,(1+\epsilon)n)$ contains a prime $p \equiv b(a)$ for $n \geq n_o$, $(a,b) = 1$.

G. Ricci [20] used the canonical factorization of the expression

$$\prod_{n=1}^{[\frac{x-b}{a}]} (na+b) \quad \text{with} \quad (a,b) = 1 \ , \quad 0 < b < a \ ,$$

to prove that the number of positive integers $na+b \leq x$ of the form np , where

$$n \leq \frac{a}{\log a + C + \sum_{p|a} \frac{\log p}{p} - 1}$$

p is prime and C is the Euler-Mascheroni constant, is greater than $\frac{c_1 x}{\log x}$ and less than $\frac{c_2 x}{\log x}$ for suitable $c_1, c_2 > 0$. In particular, this result yielded Dirichlet's theorem for the difference $a = 4,5,6,9,10,12,15,18,30$. Improving the results of his earlier paper, G. Ricci [21] proved, without the use of Dirichlet characters but with the use of the Hadamard form of the Prime Number Theorem, that in every arithmetic progression $a+b, 2a+b, \dots$, with $a > 0$, $(a,b) = 1$, there exists infinitely many numbers of the form np where p is prime and $n < \frac{a}{\log a}$. Later, P. Erdös [8] used certain combinations of expressions of the form

$$\frac{\prod_{p|a} p^{[\frac{n}{p-1}]} \prod_{k=1}^{n} (b+ka)}{n!}$$

to establish Dirichlet's theorem for $a < 29$, and, in addition, he

proved that if $\sigma = \sum_{\substack{p \nmid a \\ p<a}} \frac{1}{p} < 1$, $\lambda > \frac{a}{a-1} \cdot \frac{1}{1-\sigma}$, then the interval $(x,\lambda x]$ contains a prime $p \equiv b \pmod a$ for sufficiently large x .

In 1932, R. Breusch [3] used complex variable techniques to prove that the interval $(x,2x]$ contains a prime of each of the forms $3n \pm 1$, $4n \pm 1$ for $x \geq 7$. P. Erdös [8], in 1935, supplied an elementary proof that the interval $(x,2x]$ contains a prime of the form $6n+1$ for $x \geq 5.5$, of the form $6n+5$ for $x \geq 6.5$, and of each of the forms $4n \pm 1$ for $x \geq 3.5$. Improving these results, K. Molsen [17] used elementary methods to show that the interval $(x,\frac{9}{8} x]$ contains a prime of each of the forms $6n \pm 1$ for $x \geq 199$, and the interval $(x,\frac{4}{3} x]$ contains a prime of each of the forms $12n+d$, $d = 1,5,7,11,$ for $x \geq 118$. More recently, D. Roux [23] proved that if the integer x is sufficiently large then the set of integers $a([nx]+1)+b$, $a([nx]+2)+b$, ... , $ax+b$ contains at least one prime for $(a,b) = 1$, $(a,n) = (6,\frac{3}{4})$, $(8,\frac{1}{5})$, $(12,\frac{1}{2})$, $(30,\frac{1}{4})$. It is clear that all of these results could be improved considerably if sufficiently sharp bounds were available for $\psi(x;a,b)$.

The method that we now develop to estimate $\psi(x;a,b)$ for particular a and b stems from the approach used first by P. Tschebyschef [29] and later by other authors including J. Sylvester [27] and E. Waage [31;32] in estimating $\psi(x) = \sum_{p^{\alpha} \leq x} \log p$. We illustrate their approach by means of a simple example: Let $\Lambda(n)$ denote Mangoldt's

function, where

$$\Lambda(n) = \begin{cases} \log p & \text{if} \quad n = p^r \\ 0 & \text{otherwise} \end{cases} ,$$

then $\log n = \sum_{d|n} \Lambda(d)$ and, furthermore, $\psi(x) = \sum_{n\leq x} \Lambda(n)$. If we define $T(x) = \sum_{m\leq x} \log m$, then

$$(2.3) \qquad T(x) = \sum_{m\leq x} \sum_{d|m} \Lambda(d)$$

$$= \sum_{dn\leq x} \Lambda(d)$$

$$= \sum_{n\leq x} \psi(\tfrac{x}{n}) \quad ,$$

and, from the Möbius Inversion Formula, it follows that

$$(2.4) \qquad \psi(x) = \sum_{n\leq x} \mu(n)\, T(\tfrac{x}{n}) \quad .$$

Since there is difficulty in estimating the right-hand side of this equation directly, we consider instead $P(x,M) = \sum_{n\leq M} \mu(n)\, T(\frac{x}{n})$ and, for the purpose of this example, we choose $M = 3$. Now, from equation (2.3),

$$P(x,3) = \sum_{n\leq 3} \mu(n)\, T(\tfrac{x}{n})$$

$$= \sum_{d\leq x} a_d\, \psi(\tfrac{x}{d})$$

where $a_d = \sum_{\substack{k|d \\ k\leq 3}} \mu(k)$. If we now observe that the a_d's are periodic

with period $\prod_{p\leq 3} p = 6$, then we obtain

$$P(x,3) = \sum_{d\geq 0} \left\{ \psi(\tfrac{x}{6d+1}) + \psi(\tfrac{x}{6d+5}) - \psi(\tfrac{x}{6d+6}) \right\} .$$

Let $P'(x,3) = P(x,3) - T(\frac{x}{6})$, then

$$P'(x,3) = \psi(x) - \psi(\tfrac{x}{6}) + \sum_{d\geq 1}\left\{\psi(\tfrac{x}{6d-1}) - \psi(\tfrac{x}{6d})\right\}$$

$$+ \sum_{d\geq 1}\left\{\psi(\tfrac{x}{6d+1}) - \psi(\tfrac{x}{6d+6})\right\} .$$

Since $\psi(x)$ is a non-decreasing function of x ,

$$P'(x,3) \geq \psi(x) - \psi(\tfrac{x}{6}) \quad ,$$

and therefore

$$\sum_{\frac{\log x}{\log 6} \geq n \geq 0} P'(\tfrac{x}{6^n}, 3) \geq \psi(x) \quad .$$

The left-hand side of the above inequality can be estimated easily to obtain an upper bound for $\psi(x)$. On the other hand, since

$$P'(x,3) = \psi(x) + \psi(\tfrac{x}{5}) - \sum_{d\geq 1}\left\{\psi(\tfrac{x}{6d}) - \psi(\tfrac{x}{6d+1})\right\} - \sum_{d\geq 1}\left\{\psi(\tfrac{x}{6d}) - \psi(\tfrac{x}{6d+5})\right\}$$

$$\leq \psi(x) + \psi(\tfrac{x}{5}) ,$$

a lower bound for $\psi(x)$ can be easily computed once the upper bound is obtained.

In the estimation of $\psi(x;a,b)$ for $(a,b) = 1$ we develop equations corresponding to (2.3) and (2.4) for the functions $T(x;a,b) = \sum_{\substack{m\equiv b(a)\\ m\leq x}} \log m$ and $\psi(x;a,b)$. From the properties of $\Lambda(n)$ mentioned earlier, it is clear that

$$T(x;a,b) = \sum_{\substack{m\equiv b(a)\\ m\leq x}} \sum_{d|m} \Lambda(d)$$

(2.5)
$$= \sum_{\substack{dd'\leq x\\ dd'\equiv b(a)}} \Lambda(d')$$

$$= \sum_{\substack{(d,a)=1\\ d\leq x}} \psi(\tfrac{x}{d};a,bd^{-1}) .$$

Furthermore, since $\psi(x;a,b) = \sum_{\substack{m \equiv b(a) \\ m \leq x}} \Lambda(m)$,

$$\psi(x;a,b) = \sum_{\substack{m \equiv b(a) \\ m \leq x}} \sum_{d \mid m} \mu(d) \log \frac{m}{d}$$

$$= \sum_{\substack{dd' \equiv b(a) \\ dd' \leq x}} \mu(d) \log d' \tag{2.6}$$

$$= \sum_{\substack{(d.a)=1 \\ d \leq x}} \mu(d)\, T(\frac{x}{d}\, ;a,bd^{-1}) \ .$$

Now that we have equations (2.5) and (2.6), we can employ the method illustrated in the example to compute upper and lower bounds for $\psi(x;a,b)$ for given a and b . We do this for $a = 4$, $b = 1,3$ to establish (2.2) .

Let

$$P_1(x,10) = \sum_{\substack{(d,4)=1 \\ d \leq 10}} \mu(d)\, T(\frac{x}{d}\, ;4,d^{-1})$$

$$= \sum_{\substack{d \equiv 1(4) \\ d \leq 10}} \mu(d)\, T(\frac{x}{d}\, ;4,1) + \sum_{\substack{d \equiv 3(4) \\ d \leq 10}} \mu(d)\, T(\frac{x}{d}; 4,3) \ .$$

From (2.5),

$$T(x;4,1) = \sum_{\substack{d\equiv 1(4)\\ d\le x}} \psi(\tfrac{x}{d};4,1) + \sum_{\substack{d\equiv 3(4)\\ d\le x}} \psi(\tfrac{x}{d};4,3)$$

and

$$T(x;4,3) = \sum_{\substack{d\equiv 1(4)\\ d\le x}} \psi(\tfrac{x}{d};4,3) + \sum_{\substack{d\equiv 3(4)\\ d\le x}} \psi(\tfrac{x}{d};4,1) \quad ,$$

so that

$$P_1(x,10) = \sum_{\substack{d\equiv 1(4)\\ d\le x}} a_d\psi(\tfrac{x}{d};4,1) + \sum_{\substack{d\equiv 3(4)\\ d\le x}} a_d\psi(\tfrac{x}{d};4,3)$$

where $a_d = \sum_{\substack{k|d\\ k\le 10}} \mu(k)$. Since the a_d's have period $\prod_{p\le 10} p = 210$, the sequences $\left\{a_{4d+1}\right\}_{d\ge 0}$, $\left\{a_{4d+3}\right\}_{d\ge 0}$ have period 420 , and for $d \ge 52$,

$$a_{4d+1} = a_{4(d-52)+3}$$

$$a_{4d+3} = a_{4(d-52)+1} \quad .$$

The values of a_{4d+1}, a_{4d+3} for $0 \le d \le 52$ are given by:

$$\left\{a_{4d+1}\right\}_{d=o}^{52} = \{1,0,0,1,1,-1,0,1,0,1,1,-1,0,1,0,1,0,0,1,0,0,0,1,$$
$$0,1,1,-2,1,1,0,1,0,0,0,1,0,0,1,0,1,0,-1,1,1,0,1,$$
$$0,-1,1,1,0,0,1\} ,$$

$$\left\{a_{4d+3}\right\}_{d=o}^{51} = \{0,0,1,-1,1,1,0,1,-1,0,1,1,0,0,1,-1,1,1,-1,1,1,0,$$
$$0,0,0,1,1,0,0,0,0,1,1,-1,1,1,-1,1,0,0,1,1,0,-1,$$
$$1,0,1,1,-1,1,0,0\} .$$

Let

$$P'(x,10) = P_1(x,10) - T(\tfrac{x}{20}) - 2T(\tfrac{x}{70}) - T(\tfrac{x}{420}) \quad ,$$

then

$$P'(x,10) = \sum_{d\leq x} b_d \psi(\tfrac{x}{d}\ ;4,1) + \sum_{d\leq x} b_d \psi(\tfrac{x}{d}\ ;4,3) - \alpha_2 \log 2 \ ,$$

where

$$\alpha_2 = \sum_{2\leq 2^\alpha \leq \frac{x}{20}} \left[\frac{x}{20\cdot 2^\alpha}\right] + 2 \sum_{2\leq 2^\alpha \leq \frac{x}{70}} \left[\frac{x}{70\cdot 2^\alpha}\right] + \sum_{2\leq 2^\alpha \leq \frac{x}{420}} \left[\frac{x}{420\cdot 2^\alpha}\right] .$$

Examination of the values of the b_d's then yields the following inequalities:

(2.7) $$-\alpha_2 \log 2 + \psi(x;4,1) - \psi(\tfrac{x}{420}\ ;4,1) \leq P_1'(x,10) \ ,$$

$$(2.8)\qquad P_1'(x,10) \le \psi(x;4,1) + \{\psi(\tfrac{x}{13};4,1) + \psi(\tfrac{x}{17};4,1) + \psi(\tfrac{x}{137};4,1) - \alpha_2 \log 2 + \psi(\tfrac{x}{11};4,3) + \psi(\tfrac{x}{31};4,3) + \psi(\tfrac{x}{59};4,3) + \psi(\tfrac{x}{419};4,3)\} .$$

Repeating the above procedure for

$$P_3(x,10) = \sum_{\substack{d\equiv 1(4)\\ d\le 10}} \mu(d)\, T(\tfrac{x}{d};4,3) + \sum_{\substack{d\equiv 3(4)\\ d\le 10}} \mu(d)\, T(\tfrac{x}{d};4,1)$$

and

$$P_3'(x,10) = P_3(x,10) - T(\tfrac{x}{20}) - 2T(\tfrac{x}{70}) - T(\tfrac{x}{420})$$

in place of $P_1(x,10)$ and $P_1'(x,10)$, we obtain the analogues of inequalities (2.7) and (2.8) for $\psi(x;4,3)$:

$$(2.9)\qquad -\alpha_2 \log 2 + \psi(x;4,3) - \psi(\tfrac{x}{420};4,3) \le P_3'(x,10) ,$$

$$(2.10)\qquad P_3'(x,10) \le \psi(x;4,3) + \{\psi(\tfrac{x}{13};4,3) + \psi(\tfrac{x}{17};4,3) + \psi(\tfrac{x}{137};4,3) - \alpha_2 \log 2 + \psi(\tfrac{x}{11};4,1) + \psi(\tfrac{x}{31};4,1) + \psi(\tfrac{x}{59};4,1) + \psi(\tfrac{x}{419};4,1)\} .$$

For the purpose of estimating $P_1'(x,10)$, $P_3'(x,10)$ we use the following inequality for the gamma function [18; p. 257]:

$$0 \leq \log \Gamma(x) - (x-\tfrac{1}{2}) \log x + x - \tfrac{1}{2} \log 2\pi \leq \frac{1}{12x} , \quad x > 0 .$$

Suppressing the details, this inequality yields the following inequalities in which K and L are given by

$$K = \frac{1}{4} \log 4 + \log \sqrt{2\pi} - \log \Gamma (\tfrac{1}{4}) ,$$

and

$$L = - \frac{1}{4} \log 4 + \log \sqrt{2\pi} - \log \Gamma (\tfrac{3}{4}) :$$

$$(2.11) \quad - \frac{1}{2} \log x \leq T(x) - x \log x + x - \log \sqrt{2\pi} \leq \frac{1}{2} \log x + \frac{7}{12x} , \quad x \geq \frac{3}{2} ;$$

$$(2.12) \quad - \frac{1}{4} \log x - \frac{3}{8x} \leq T(x;4,1) - \frac{x}{4} \log x + \frac{x}{4} + K \leq \frac{1}{2} \log x + \frac{7}{3x} , \quad x \geq 11 ;$$

$$(2.13) \quad - \frac{1}{4} \log x - \frac{3}{8x} \leq T(x;4,3) - \frac{x}{4} \log x + \frac{x}{4} + L \leq \frac{1}{2} \log x + \frac{1}{3x} , \quad x \geq 11 .$$

Thus, from (2.7), (2.9) and the above inequalities,

$$(2.14) \quad \psi(x;4,1) - \psi(\tfrac{x}{420};4,1) \leq .58317x + \frac{13}{4} \log x - 14.34291 + \frac{191}{24x}$$

$$(2.15) \quad \psi(x;4,3) - \psi(\tfrac{x}{420};4,3) \leq .58317x + \frac{13}{4} \log x - 13.58309 + \frac{143}{24x}$$

for $x \geq 630$. Since $\psi(x;4,b) \leq \psi(x)$ for $b = 1,3$, we use the inequality

$$\psi(x) < 1.03883x \qquad \text{for} \quad x > 0$$

of J. Rosser and L. Schoenfeld [22]. Hence, from inequalities (2.14)

and (2.15) ,

$$\psi(x;4,b) < .59x$$

for $b = 1,3$, $x \geq 6000$.

In order to check the above inequality for $x < 6000$ it was found necessary to compute $\psi(x;4,b)$, $b = 1,3$, for $x \leq 600$ and to estimate roughly these functions for $600 < x < 6000$. The table of values of $\psi(x;4,1)$ and $\psi(x;4,3)$ for $x \leq 600$ will be found in the Appendix. As a result of the computations performed, we have

$$\psi(x;4,b) < .59x \tag{2.16}$$

for $b = 1,3$, $x > 0$.

Now, from inequalities (2.8), (2.10), (2.11), (2.12), (2.13), we obtain

$$\psi(x;4,1) + f_1(x) \geq .58317x - \frac{31}{4}\log x + 23.27185 - \frac{8489}{24x} , \tag{2.17}$$

$$\psi(x;4,3) + f_3(x) \geq .58317x - \frac{31}{4}\log x + 24.05537 - \frac{8729}{24x} , \tag{2.18}$$

for $x \geq 630$, where

$$f_1(x) = \psi(\tfrac{x}{13};4,1) + \psi(\tfrac{x}{17};4,1) + \psi(\tfrac{x}{137};4,1) + \psi(\tfrac{x}{11};4,3)$$
$$+ \psi(\tfrac{x}{31};4,3) + \psi(\tfrac{x}{59};4,3) + \psi(\tfrac{x}{419};4,3) \quad ,$$

$$f_2(x) = \psi(\tfrac{x}{13};4,3) + \psi(\tfrac{x}{17};4,3) + \psi(\tfrac{x}{137};4,3) + \psi(\tfrac{x}{11};4,1)$$

$$+ \psi(\tfrac{x}{31};4,1) + \psi(\tfrac{x}{59};4,1) + \psi(\tfrac{x}{419};4,1) \ .$$

Thus, from (2.16) we obtain

$$f_i(x) \le .16848x \ , \quad i = 1,2,$$

which, together with (2.17) and (2.18), yields

$$\psi(x;4,b) > .40x$$

for $b = 1,3, \quad x \ge 6000$.

To verify the above inequality for $x \le 600$, we referred to the table of values of $\psi(x;4,1)$ and $\psi(x;4,3)$ in the Appendix. In addition, we estimated these function values for $600 < x < 6000$ and found the above inequality to hold for x in this range. We therefore have the following:

$$\psi(x;4,1) > .40x \quad \text{for} \quad x \ge 37 \ ,$$

and

$$\psi(x;4,3) > .40x \quad \text{for} \quad x \ge 43 \ .$$

CHAPTER III

TWO ELEMENTARY PROOFS OF A SUM INVOLVING PRIMES OF THE FORM $4n + 1$

It follows from the Prime Number Theorem for arithmetic progressions that

$$(3.1) \qquad \sum_{\substack{p \equiv b \pmod{a} \\ p \leq x}} \frac{\log p}{p} = \frac{1}{\phi(a)} \log x + O(1) ,$$

where p is prime, ϕ is Euler's totient function, a and b are positive integers with $(a,b) = 1$, and the constant implied by $O(1)$ is dependent upon a and b . It is, however, of interest to find elementary proofs of such relations. In this chapter we intend to give two elementary proofs of (3.1) for the special case $a = 4$, $b = 1$. While obvious minor variations in the two proofs yield (3.1) for $a = 3$, $b = 1$, the methods do not seem to apply if $\phi(a) > 2$.

R. Breusch [4] supplied an elementary proof of (3.1) for $a = 4$, $b = 1$ by analysing $\prod_{r=1}^{n} \prod_{s=1}^{n} (r^2 + s^2)$. However, estimating the contribution of the various primes in the canonical factorization of this expression presented considerable difficulty. A somewhat more natural expression to consider seems to be

$$P(n) = \prod_{\substack{1 \le r^2+s^2 \le n \\ r,s \text{ integers}}} (r^2 + s^2)$$

and it is this expression which we now use to establish (3.1) for $a = 4$, $b = 1$.

Let $r(n)$ denote the number of representations of n in the form $x^2 + y^2$, where x and y are integers. For the sake of simplicity of notation we introduce the characters modulo 4 : Let χ_o and χ represent the principal and non-principal characters modulo 4, respectively; then the following results are well-known (see, for example [14]):

$$r(m) = 4 \sum_{d|m} \chi(d) , \quad m > 0 , \quad m \text{ an integer},$$

$$A(n) = \sum_{1 \le m \le n} r(m)$$

$$= 4 \sum_{1 \le d \le n} \chi(d) \left[\frac{n}{d}\right] .$$

Observe that $A(n)$ is the number of lattice-points, excluding the origin, in the circle centred at the origin with radius $\sqrt{n}$, that is,

$$A(n) = \pi n + O(\sqrt{n}) . \tag{3.2}$$

Now consider

$$\log P(n) = \sum_{\substack{1\le r^2+s^2\le n \\ r,s \text{ integers}}} \log (r^2 + s^2)$$

$$= \sum_{1\le m\le n} r(m) \log m$$

(3.3)

$$= - \int_1^n \frac{A(t)}{t}\, dt + A(n) \log n$$

$$= \pi n \log n - \pi n + O(\sqrt{n} \log n) \quad ,$$

where we have used the formula for integration by parts for the Stieltjes integral. In order to determine explicitly the canonical factorization of $P(n)$, let

$$P(n) = 2^{\alpha_2(n)} \prod p^{\alpha_p(n)} \prod q^{\alpha_q(n)} \quad ,$$

where p and q denote primes $\equiv 1$ and $3 \pmod 4$, respectively. Since $\sum_{d|m} \chi(d)$ is a (weakly) multiplicative function of m and $\chi(p^\alpha) = 1$ for $\alpha = 0,1,2,\ldots,$ it follows that

$$\alpha_p = \alpha_p(n) = \sum_{\substack{1\le m\le n \\ p\|m}} r(m) + 2 \sum_{\substack{1\le m\le n \\ p^2\|m}} r(m) + 3 \sum_{\substack{1\le m\le n \\ p^3\|m}} r(m) + \ldots$$

$$= 4 \sum_{j \geq 1} j \sum_{\substack{1 \leq m \leq n \\ p^j \| m}} \sum_{d \mid m} \chi(d)$$

$$= 4 \sum_{j \geq 1} \sum_{\substack{1 \leq m \leq n \\ p^j \| m}} \Big(\sum_{d \mid p^j} \chi(d) \Big) \Big(\sum_{d \mid \frac{m}{p^j}} \chi(d) \Big)$$

$$= 4 \sum_{j \geq 1} j \sum_{\substack{1 \leq m \leq n \\ p^j \| m}} \left(\frac{j+1}{j}\right) \Big(\sum_{d \mid p^{j-1}} \chi(d) \Big) \Big(\sum_{d \mid \frac{m}{p^j}} \chi(d) \Big)$$

$$= 4 \sum_{j \geq 1} (j+1) \sum_{\substack{1 \leq m \leq \frac{n}{p} \\ p^{j-1} \| m}} \sum_{d \mid m} \chi(d)$$

$$= 4 \Big(2 \sum_{1 \leq m \leq \frac{n}{p}} \sum_{d \mid m} \chi(d) + \sum_{j \geq 2} (j-1) \sum_{\substack{1 \leq m \leq \frac{n}{p} \\ p^{j-1} \| m}} \sum_{d \mid m} \chi(d) \Big)$$

$$= 2 \, A\left(\frac{n}{p}\right) + \alpha_p\left(\frac{n}{p}\right) \quad .$$

Thus,

$$\alpha_p = 2 \left(A\left(\frac{n}{p}\right) + A\left(\frac{n}{p^2}\right) + A\left(\frac{n}{p^3}\right) + \cdots \right) . \tag{3.4}$$

Similarly, since $\chi(q^\alpha) = (-1)^\alpha$ for $\alpha = 0,1,2,\ldots$,

$$\alpha_q = \alpha_q(n) = \sum_{j\geq 1} j \sum_{\substack{1\leq m\leq n \\ q^j \| m}} r(m)$$

$$= 4 \sum_{j\geq 1} j \sum_{\substack{1\leq m\leq n \\ q^j \| m}} \sum_{d|m} \chi(d)$$

$$= 4 \sum_{j\geq 1} j \sum_{\substack{1\leq m\leq n \\ q^j \| m}} \Big(\sum_{d|q^j} \chi(d) \Big) \Big(\sum_{d|\frac{m}{q^j}} \chi(d) \Big)$$

$$= 4 \sum_{j\geq 1} 2j \sum_{\substack{1\leq m\leq n \\ q^{2j} \| m}} \Big(\sum_{d|q^{2j}} \chi(d) \Big) \Big(\sum_{d|\frac{m}{q^{2j}}} \chi(d) \Big)$$

$$= 4 \sum_{j\geq 1} 2j \sum_{\substack{1\leq m\leq n \\ q^{2j} \| m}} \Big(\sum_{d|q^{2j-2}} \chi(d) \Big) \Big(\sum_{d|\frac{m}{q^{2j}}} \chi(d) \Big)$$

$$= 4 \sum_{j\geq 1} 2j \sum_{\substack{1\leq m\leq \frac{n}{q^2} \\ q^{2j-2} \| m}} \sum_{d|m} \chi(d)$$

$$= 4 \Bigg(2 \sum_{1\leq m\leq \frac{n}{q^2}} \sum_{d|m} \chi(d) + \sum_{j\geq 2} (2j-2) \sum_{\substack{1\leq m\leq \frac{n}{q^2} \\ q^{2j-2} \| m}} \sum_{d|m} \chi(d) \Bigg)$$

$$= 2\,A\Big(\frac{n}{q^2}\Big) + \alpha_q\Big(\frac{n}{q^2}\Big).$$

It follows that

$$\alpha_q = 2\left(A\left(\frac{n}{q^2}\right) + A\left(\frac{n}{q^4}\right) + A\left(\frac{n}{q^6}\right) + \cdots\right) . \tag{3.5}$$

Now, since $\chi(2^\alpha) = 0$ for $\alpha = 1,2,3,\ldots$,

$$\alpha_2 = \alpha_2(n) = \sum_{j\geq 1} j \sum_{\substack{1\leq m\leq n \\ 2^j \| m}} r(m)$$

$$= 4 \sum_{j\geq 1} \sum_{\substack{1\leq m\leq n \\ 2^j \| m}} \sum_{d \mid m} \chi(d)$$

$$= 4 \sum_{j\geq 1} j \sum_{\substack{1\leq m\leq n \\ 2^j \| m}} \left(\sum_{d \mid 2^j} \chi(d)\right)\left(\sum_{d \mid \frac{m}{2^j}} \chi(d)\right)$$

$$= 4 \sum_{j\geq 1} j \sum_{\substack{1\leq m\leq n \\ 2^j \| m}} \left(\sum_{d \mid 2^{j-1}} \chi(d)\right)\left(\sum_{d \mid \frac{m}{2^j}} \chi(d)\right)$$

$$= 4 \sum_{j\geq 1} j \sum_{\substack{1\leq m\leq \frac{n}{2} \\ 2^{j-1} \| m}} \sum_{d \mid m} \chi(d)$$

$$= 4\left(\sum_{1\leq m\leq \frac{n}{2}} \sum_{d \mid m} \chi(d) + \sum_{j\geq 2} (j-1) \sum_{\substack{1\leq m\leq \frac{n}{2} \\ 2^{j-1} \| m}} \sum_{d \mid m} \chi(d)\right)$$

$$= A(\tfrac{n}{2}) + \alpha_2(\tfrac{n}{2}) \quad .$$

Hence

$$\alpha_2 = A(\tfrac{n}{2}) + A\left(\frac{n}{2^2}\right) + A\left(\frac{n}{2^3}\right) + \dots \quad . \tag{3.6}$$

From the interpretation of $A(n)$ as the number of lattice-points, excluding the origin, in the circle centred at the origin with radius $\sqrt{n}$ and from equations (3.4), (3.5), it is clear that $\alpha_p = 0$ for $p > n$ and $\alpha_q = 0$ for $q > \sqrt{n}$, while $\alpha_p > 0$ and $\alpha_q > 0$ if $p \leq n$ and $q \leq \sqrt{n}$, respectively. Thus, from (3.4), (3.5), and (3.6),

$$\log P(n) = \alpha_2 \log 2 + \sum_{p \leq n} \alpha_p \log p + \sum_{q \leq \sqrt{n}} \alpha_q \log q$$

$$= (A(\tfrac{n}{2}) + A\left(\frac{n}{2^2}\right) + \dots)\log 2 + 2\sum_{p \leq n}(A(\tfrac{n}{p}) + A\left(\frac{n}{p^2}\right) + \dots)\log p \tag{3.7}$$

$$+ 2\sum_{q \leq \sqrt{n}}\left(A\left(\frac{n}{q^2}\right) + A\left(\frac{n}{q^4}\right) + \dots\right)\log q \quad .$$

To determine the magnitude of the three sums on the right-hand side of equation (3.7), we use (3.2):

$$\alpha_p = 2\pi\left(\frac{n}{p} + \frac{n}{p^2} + \cdots\right) + O(\sqrt{n}) \sum_{m\geq 1} \frac{1}{p^{m/2}}$$

$$= 2\pi\,\frac{n}{p} + O\left(\frac{n}{p^2}\right) + O\left(\sqrt{\frac{n}{p}}\right) \quad ;$$

thus

$$\sum_{p\leq n} \alpha_p \log p = 2\pi n \sum_{p\leq n} \frac{\log p}{p} + O(n) \sum_{p\leq n} \frac{\log p}{p^2} + O(\sqrt{n}) \sum_{p\leq n} \frac{\log p}{\sqrt{p}} \quad .$$

Similarly,

$$\sum_{q\leq\sqrt{n}} \alpha_q \log q = O(n) \sum_{q\leq\sqrt{n}} \frac{\log q}{q^2} + O(\sqrt{n}) \sum_{q\leq\sqrt{n}} \frac{\log q}{\sqrt{q}} \quad ,$$

and

$$\alpha_2 \log 2 = O(n) \quad .$$

Using partial summation and the result that

$$\theta(x) = \sum_{\substack{r\leq x \\ r \text{ prime}}} \log r = O(x) \ ,$$

we obtain the estimates

$$\sum_{\substack{r\leq x \\ r \text{ prime}}} \frac{\log r}{\sqrt{r}} = O(\sqrt{x}) \quad \text{and} \quad \sum_{\substack{r\leq x \\ r \text{ prime}}} \frac{\log r}{r} = O(\log x) \ .$$

In addition, $\sum_{m\geq 1} \frac{\log m}{m^2} = O(1)$ so that

$$\sum_{p\leq n} \alpha_p \log p = 2\pi n \sum_{p\leq n} \frac{\log p}{p} + O(n)$$

$$\sum_{q\leq\sqrt{n}} \alpha_q \log q = O(n) \quad .$$

The above results together with (3.3) yield

$$\pi n \log n = 2\pi n \sum_{p\leq n} \frac{\log p}{p} + O(n) \quad .$$

Hence

$$\sum_{\substack{p\equiv 1(4)\\ p\leq n}} \frac{\log p}{p} = \frac{1}{2} \log n + O(1) \quad .$$

The second proof we give of (3.1) for $a = 4$, $b = 1$ is of a completely different nature and has the advantage of providing a good estimate of the constant term implied by the $O(1)$ in (3.1). An explicit upper bound will be obtained for this constant in the next chapter, but, for the present, the objective is merely to give a short elementary proof of (3.1).

Let $\Lambda(n)$ denote Mangoldt's function, then from the definition of the principal and non-principal characters, χ_o and χ, modulo 4, it is clear that

$$f(n) = \sum_{\substack{m \equiv 1(4) \\ m \leq n}} \frac{\Lambda(m)}{m}$$

$$= \frac{1}{2} \sum_{m \leq n} \frac{\chi_o(m)\,\Lambda(m)}{m} + \frac{1}{2} \sum_{m \leq n} \frac{\chi(m)\,\Lambda(m)}{m}$$

$$= \frac{1}{2} \sum_{m \leq n} \frac{\Lambda(m)}{m} - \frac{\log 2}{2} \sum_{2 \leq 2^{\alpha} \leq n} \frac{1}{2^{\alpha}} + \frac{1}{2} \sum_{m \leq n} \frac{\chi(m)\,\Lambda(m)}{m} .$$

Now consider

$$\sum_{u \leq n} \frac{\chi(u)}{u} f\left(\frac{n}{u}\right) = \sum_{\substack{m \equiv 1(4) \\ m \leq n}} \frac{\Lambda(m)}{m} \sum_{u \leq \frac{n}{m}} \frac{\chi(u)}{u} .$$

Since

$$\sum_{u \leq x} \frac{\chi(u)}{u} = \frac{\pi}{4} + O\left(\frac{1}{x}\right)$$

it follows that

$$\sum_{u \leq n} \frac{\chi(u)}{u} f\left(\frac{n}{u}\right) = \frac{\pi}{4} \sum_{\substack{m \equiv 1(4) \\ m \leq n}} \frac{\Lambda(m)}{m} + O\left(\frac{1}{n}\right) \sum_{\substack{m \equiv 1(4) \\ m \leq n}} \Lambda(m) . \tag{3.8}$$

Similarly,

$$\sum_{u \leq n} \frac{\chi(u)}{u} \sum_{m \leq \frac{n}{u}} \frac{\Lambda(m)}{m} = \frac{\pi}{4} \sum_{m \leq n} \frac{\Lambda(m)}{m} + O\left(\frac{1}{n}\right) \sum_{m \leq n} \Lambda(m) \tag{3.9}$$

and

$$(3.10) \qquad \sum_{u\le n} \frac{\chi(u)}{u} \sum_{2\le 2^{\alpha}\le \frac{n}{u}} \frac{1}{2^{\alpha}} = \frac{\pi}{4} \sum_{2\le 2^{\alpha}\le n} \frac{1}{2^{\alpha}} + O(\tfrac{1}{n}) \sum_{2\le 2^{\alpha}\le n} 1$$

$$= O(1) \quad .$$

Finally, since $\sum_{d|n} \Lambda(d) = \log n$,

$$(3.11) \qquad \sum_{u\le n} \frac{\chi(u)}{u} \sum_{m\le \frac{n}{u}} \frac{\chi(m)\,\Lambda(m)}{m} = \sum_{s\le n} \frac{\chi(s)}{s} \sum_{m|s} \Lambda(m)$$

$$= \sum_{s\le n} \frac{\chi(s)\log s}{s}$$

$$= O(1) \quad .$$

Among the first estimates in the Tschebyschef Theory of the Distribution of Primes are:

$$\sum_{p\le x} \frac{\log p}{p} = \log x + O(1)$$

and

$$\psi(x) = \sum_{p^{\alpha}\le x} \log p$$

$$= O(x) \quad ;$$

thus, with these estimates and equations (3.8), (3.9), (3.10), and (3.11) we obtain

$$(3.12) \qquad \frac{\pi}{4} \sum_{\substack{m \equiv 1(4) \\ m \leq n}} \frac{\Lambda(m)}{m} = \frac{\pi}{8} \sum_{m \leq n} \frac{\Lambda(m)}{m} + O(1) \quad .$$

Since

$$\sum_{\substack{m \equiv 1(4) \\ m \leq n}} \frac{\Lambda(m)}{m} = \sum_{\substack{p \equiv 1(4) \\ p \leq n}} \frac{\log p}{p} + \sum_{\substack{r \geq 2 \\ p^r \equiv 1(4) \\ p^r \leq n}} \frac{\log p}{p^r}$$

$$= \sum_{\substack{p \equiv 1(4) \\ p \leq n}} \frac{\log p}{p} + O(1) \quad ,$$

it follows from (3.12) that

$$\sum_{\substack{p \equiv 1(4) \\ p \leq n}} \frac{\log p}{p} = \frac{1}{2} \log n + O(1) \quad .$$

CHAPTER IV

ON THE DIOPHANTINE EQUATION $n! = x^4 - y^4$

Following the method used in the latter part of Chapter III we will prove that

$$(4.1)\qquad \sum_{\substack{p\equiv 1(4)\\ p^r \leq n\\ 1\leq r}} \frac{\log p}{p^r} < \frac{1}{2}\log n - .59571 + \frac{.91}{\log n} + \frac{4}{\sqrt{n}}\,, \quad \text{for} \quad n \geq 961\,,$$

and use this result to complete the proof of the insolvability of the diophantine equation $n! = x^4 - y^4$, $(x,y) = 1$, studied by P. Erdős and R. Obláth [10].

Let χ_o and χ represent the principal and non-principal characters modulo 4 , respectively. Further, let $f(x)$ be determined by

$$(4.2)\qquad \sum_{u\leq x} \frac{\chi(u)}{u} = \sum_{1\leq 2m+1\leq x} \frac{(-1)^m}{2m+1} = \frac{\pi}{4} + f(x)\quad ,$$

then $|f(x)| < \frac{1}{x}$, $x > 0$. As we saw in Chapter III ,

$$\sum_{\substack{m\equiv 1(4)\\ m\leq n}} \frac{\Lambda(m)}{m} = \frac{1}{2}\sum_{m\leq n}\frac{\Lambda(m)}{m} - \frac{\log 2}{2}\sum_{2\leq 2^\alpha\leq n}\frac{1}{2^\alpha} + \frac{1}{2}\sum_{m\leq n}\frac{\chi(m)\,\Lambda(m)}{m}\;;$$

thus, from (4.2),

$$(4.3)\qquad \sum_{\substack{m\equiv 1(4)\\ m\leq n}} \frac{\Lambda(m)}{m} = \frac{1}{2}\sum_{m\leq n}\frac{\Lambda(m)}{m} - \frac{2}{\pi}\sum_{m\leq n}\frac{\chi(m)\,\Lambda(m)\,f(n/m)}{m} - \frac{\log 2}{2}\sum_{2\leq 2^{\alpha}\leq n}\frac{1}{2^{\alpha}}$$

$$-\frac{2}{\pi}\log 2\sum_{2\leq 2^{\alpha}\leq n}\frac{1}{2^{\alpha}}\,f\left(\frac{n}{2^{\alpha}}\right) + \frac{2}{\pi}\sum_{s\leq n}\frac{\chi(s)\log s}{s}\ .$$

The greatest difficulty in the estimation of the terms on the right-hand side of (4.3) is presented by the expression

$$-\sum_{m\leq n}\frac{\chi(m)\,\Lambda(m)\,f(n/m)}{m}\quad ,$$

and we will now determine an upper bound for it.

Let $K(x) = \psi(x;4,3) - \psi(x;4,1)$, then, from inequality (2.2), we have $|K(x)| < .19x$ for $x \geq 43$. Direct checking of the table of $\psi(x;4,1)$ and $\psi(x;4,3)$ in the Appendix for $x < 43$ gives

$$(4.4)\qquad |K(x)| < .19x \qquad \text{for} \quad x \geq 13\ .$$

Define

$$S(r,s) = \sum_{s<m\leq r}\frac{K(m) - K(m-1)}{m}\,f(\tfrac{n}{m})\quad ;$$

then from (4.2),

$$S(n,\tfrac{n}{3}) = f(1) \sum_{\frac{n}{3}<m\le n} \frac{K(m) - K(m-1)}{m} ,$$

$$S(\tfrac{n}{3},\tfrac{n}{5}) = f(3) \sum_{\frac{n}{5}<m\le \frac{n}{3}} \frac{K(m) - K(m-1)}{m} ,$$

$$\vdots$$

$$S(\tfrac{n}{15},\tfrac{n}{17}) = f(15) \sum_{\frac{n}{17}<m\le \frac{n}{15}} \frac{K(m) - K(m-1)}{m} .$$

Furthermore, since

$$\sum_{s<m\le r} \frac{K(m) - K(m-1)}{m} = \sum_{s<m\le r} \frac{K(m)}{m(m+1)} - \frac{K(s)}{[s]+1} + \frac{K(r)}{[r]+1}$$

we have

$$-\sum_{m\le n} \frac{\chi(m)\Lambda(m)}{m} f(\tfrac{n}{m}) = \sum_{t=1}^{8} f(2t-1) \sum_{\frac{n}{2t+1}<m\le \frac{n}{2t-1}} \frac{K(m)}{m(m+1)} + \sum_{t=1}^{8} f(2t-1) \left\{ \frac{-K(\frac{n}{2t+1})}{[\frac{n}{2t+1}]+1} + \frac{K(\frac{n}{2t-1})}{[\frac{n}{2t-1}]+1} \right\}$$

$$+ \; S(\tfrac{n}{17},1)$$

$$= t_1 + t_2 + S(\tfrac{n}{17},1) .$$

Now

$$\sum_{j=1}^{n} \frac{1}{j} = \log n + C + \frac{1}{2n} + \frac{\theta}{8n^2} \quad , \qquad 0 < \theta < 1 \quad ,$$

so that

$$\sum_{s<m\leq r} \frac{1}{m+1} \leq \log (r+1) - \log s + \frac{1}{2r} - \frac{1}{2(s+1)} + \frac{1}{8r^2} \quad .$$

Thus for $n \geq 221$, (4.4) implies that

$$|t_1| \leq .19 \sum_{t=1}^{8} |f(2t-1)| \sum_{\frac{n}{2t+1} < m \leq \frac{n}{2t-1}} \frac{1}{m+1}$$

$$\leq .19 \sum_{t=1}^{8} |f(2t-1)| \left\{ \log \frac{2t+1}{2t-1} + \frac{3(2t-1)}{2n} - \frac{(2t+1)}{2(n+2t+1)} + \frac{(2t-1)^2}{8n^2} \right\} .$$

Since

$$\begin{aligned} f(1) &= .21464 & \qquad f(9) &= .04957 \\ f(3) &= -.11869 & f(11) &= -.04134 \\ f(5) &= .08131 & f(13) &= .03558 \\ f(7) &= -.06154 & f(15) &= -.03109 \quad , \end{aligned}$$

it follows that

$$|t_1| \leq .06938 + \frac{17}{n} + \frac{373}{n^2} \qquad \text{for} \quad n \geq 221 \quad .$$

From the definiton of $K(x)$, we have

$$K(x) = \psi(x) - [\frac{\log x}{\log 2}] \log 2 - 2\psi(x;4,1) \quad ;$$

hence

$$t_2 = f(1)\,\frac{K(n)}{[n]+1} - f(15)\,\frac{K(n/17)}{[n/17]+1} + (f(3) - f(1))\,\frac{K(n/3)}{[n/3]+1}$$

$$+\sum_{t=2}^{7} \frac{(-1)^t}{2t+1} \left\{ \frac{\psi(\frac{n}{2t+1}) - [\frac{\log n/2t+1}{\log 2}] \log 2}{[\frac{n}{2t+1}]+1} \right\} -2\sum_{t=2}^{7} (-1)^t \,\frac{\psi(\frac{n}{2t+1};4,1)}{[\frac{n}{2t+1}]+1}$$

$$\leq f(1)\,\frac{K(n)}{[n]+1} - f(15)\,\frac{K(\frac{n}{17})}{[\frac{n}{17}]+1} - \frac{1}{3}\,\frac{K(\frac{n}{3})}{[\frac{n}{3}]+1} + \frac{1}{n}\sum_{t=2}^{7} (-1)^t \,\psi(\frac{n}{2t+1}) + E(n) \;,$$

where

$$E(n) \leq \frac{1}{n^2} \left\{ 7\,\psi(\frac{n}{7}) + 11\,\psi(\frac{n}{11}) + 15\,\psi(\frac{n}{15}) \right\}$$

$$+ \frac{2}{n^2} \left\{ 5\,\psi(\frac{n}{5};4,1) + 9\,\psi(\frac{n}{9};4,1) + 13\,\psi(\frac{n}{13};4,1) \right\} + \frac{2 \log 2}{n} \;.$$

J. Rosser and L. Schoenfeld [22] have shown that

$$\psi(x) < \theta(x) + 1.42620\sqrt{x}$$

(4.5) $$< x + \frac{x}{2 \log x} + 1.42620\sqrt{x} \qquad \text{for} \quad x > 1 \;,$$

$$x - \frac{x}{\log x} < \theta(x) \leq \psi(x) \qquad \text{for} \quad x \geq 41 \;,$$

and

$$(4.6) \qquad \psi(x) < 1.03883x \qquad \text{for} \quad x > 0 \ .$$

For $x < 41$, easy computation extends their estimate above to

$$(4.7) \qquad x - \frac{x}{\log x} < \psi(x) \qquad \text{for} \quad x \geq 6 \ .$$

For convenience, we use inequality (4.6) along with (2.2) to estimate $E(n)$; thus

$$E(n) < \frac{9}{n} \ .$$

From inequalities (4.5) and (4.7) we obtain

$$\frac{1}{n} \sum_{t=2}^{7} (-1)^t \psi\left(\frac{n}{2t+1}\right) < .08764 + \frac{.51}{\log n/15} + \frac{9}{2\sqrt{5n}} \qquad \text{for} \quad n \geq 90 \ ,$$

while inequality (4.4) gives

$$\frac{|f(1)\ K(n)|}{[n]+1} + \frac{|f(15)\ K(\frac{n}{17})|}{[\frac{n}{17}]+1} + \frac{1}{3}\,\frac{|K(\frac{n}{3})|}{[\frac{n}{3}]+1} < .11004 \qquad \text{for} \quad n \geq 221 \ .$$

Hence

$$t_2 \leq .19768 + \frac{.51}{\log n/15} + \frac{9}{2\sqrt{5n}} + \frac{9}{n} \qquad \text{for} \quad n \geq 221 \ .$$

Collecting results up to this point, we have

$$-\sum_{m\leq n}\frac{\chi(m)\Lambda(m)}{m}\,f(\tfrac{n}{m}) < .26706 + \frac{.51}{\log n/15} + \frac{9}{2\sqrt{5n}} + \frac{26}{n} + \frac{373}{n^2} + S(\tfrac{n}{17},1)$$

for $n \geq 221$. Since

$$S(\tfrac{n}{17},1) = -\sum_{m\leq\frac{n}{17}}\frac{\chi(m)\Lambda(m)}{m}\,f(\tfrac{n}{m})$$

it follows from (4.6) and $|f(x)| < \frac{1}{x}$ for $x > 0$ that

$$\begin{aligned}|S(\tfrac{n}{17},1)| &\leq \sum_{m\leq\frac{n}{17}}\frac{\Lambda(m)}{m}\,|f(\tfrac{n}{m})| \\ &< \frac{1}{n}\,\psi(\tfrac{n}{17}) \\ &< .06111 \quad .\end{aligned}$$

Thus, for $n \geq 221$,

$$-\sum_{m\leq n}\frac{\chi(m)\Lambda(m)f(\frac{n}{m})}{m} < .32817 + \frac{.51}{\log n/15} + \frac{4}{\sqrt{n}} \quad . \tag{4.8}$$

Referring once again to expression (4.3),

$$\sum_{s\leq n}\frac{\chi(s)\log s}{s} \leq \sum_{s\leq 49}\frac{\chi(s)\log s}{s} < -.15367 \qquad \text{for} \quad n \geq 49 \quad .$$

$$-\frac{\log 2}{2}\sum_{2\leq 2^\alpha\leq n}\frac{1}{2^\alpha} < -\frac{\log 2}{2} + \frac{\log 2}{n} ,$$

and

$$-\frac{2}{\pi}\log 2 \sum_{2\leq 2^{\alpha}\leq n} \frac{1}{2^{\alpha}}\, f\left(\frac{n}{2^{\alpha}}\right) < \frac{2}{\pi}\,\frac{\log n}{n} \ .$$

Hence, from (4.3), (4.8), and the above inequalities,

$$\sum_{\substack{m\equiv 1(4)\\ m\leq n}} \frac{\Lambda(m)}{m} \leq \frac{1}{2}\sum_{m\leq n}\frac{\Lambda(m)}{m} - .23548 + \frac{.33}{\log n/15} + \frac{4}{\sqrt{n}}$$

for $n \geq 221$.

Now J. Rosser and L. Schoenfeld [22] have shown that

$$(4.9) \qquad \sum_{p\leq x} \frac{\log p}{p} < \log x + E + \frac{1}{2\log x} \qquad \text{for} \quad x \geq 319 \ ,$$

where $E < -1.33258$, and from their evaluations of $\sum_{p} \frac{\log p}{p^r}$ for $2 \leq r \leq 29$ it follows that

$$(4.10) \qquad \sum_{r\geq 2}\ \sum_{p} \frac{\log p}{p^r} < .75543 \ .$$

Thus from (4.9) and (4.10),

$$(4.11) \qquad \sum_{\substack{p\equiv 1(4)\\ p^r\leq n\\ r\geq 1}} \frac{\log p}{p^r} < \frac{1}{2}\sum_{p\leq n}\frac{\log p}{p} + \frac{1}{2}\left\{\sum_{\substack{p^r\leq n\\ r\geq 2}}\frac{\log p}{p^r} - 2\sum_{\substack{p\equiv 3(4)\\ p^{2r}\leq n\\ r\geq 1}}\frac{\log p}{p^{2r}}\right\} - .23548 +$$

$$\frac{.33}{\log n/15} + \frac{4}{\sqrt{n}}$$

$$< \frac{1}{2}\log n + \frac{1}{2}E + .07058 + \frac{.33}{\log n/15} + \frac{1}{4\log n} + \frac{4}{\sqrt{n}}$$

for $n \geq 961$. Hence

$$\sum_{\substack{p \equiv 1(4) \\ p^r \leq n \\ r \geq 1}} \frac{\log p}{p^r} < \frac{1}{2}\log n - .59571 + \frac{.91}{\log n} + \frac{4}{\sqrt{n}}$$

for $n \geq 961$, which gives (4.1) . We now use this result to settle the question of the solvability of the diophantine equation $n! = x^4 - y^4$, $(x,y) = 1$.

The motivation for studying diophantine equations of this form stems from the conjecture concerning the solvability of $n! = x^2 - 1$. In 1876, H. Brocard [5] asked for what values of n is $n! + 1$ a square, and later, [6], he formulated the conjecture that $n! + 1$ is a square only if $n = 4, 5,$ or 7 . Several attempts have been made to settle this conjecture but relatively little success has been obtained. M. Kraitchik [15] proved that $n! + 1 \nmid x^2$ for $7 < n < 1020$. Seemingly unaware of Kraitchik's results, H. Gupta [13] proved that if $n! + 1 = x^2$ is to hold for $n > 7$ then $n > 50$ and $n!$ must have more than 34 digits. The author [11], with the aid of the I.B.M. 1620 at the University of Alberta, extended Kraitchik's results to show that

$n! + 1 \neq x^2$ for $7 < n \leq 10{,}000$. This was accomplished by examining the quadratic character of $n! + 1$ modulo p for $n \leq 10{,}000$ and $10{,}007 \leq p \leq 10{,}111$, where p denotes a prime.

P. Erdős and R. Obláth [10] considered the diophantine equations $n! = x^m - y^m$, $m > 2$, $(x,y) = 1$, and completely settled the question of their solvability for $m \neq 4$. In the case of $n! = x^4 - y^4$, they were able to prove that this equation is insolvable for $n > n_o$ - however, the n_o was not determined. Using essentially their approach to the problem together with inequality (4.1), we can now prove that $n! = x^4 - y^4$ has no solutions in positive integers n,x,y with $(x,y) = 1$.

For $b = 1,3,$ let

$$S(n;4,b) = \sum_{\substack{p \equiv b(4) \\ p^r \leq n \\ r \geq 1}} \log p \left[\frac{n}{p^r} \right]$$

and let

$$S(n,2) = \sum_{\substack{2^r \leq n \\ r \geq 1}} \log 2 \left[\frac{n}{2^r} \right] ;$$

then

$$\log n! = S(n,2) + S(n;4,1) + S(n;4,3) \quad . \tag{4.12}$$

Now if $n! = (x^2-y^2)(x^2+y^2)$ is to hold for some integer n with $(x,y) = 1$, the factor (x^2-y^2) must contain $\exp(S(n;4,3))$ together with at least $\exp(S(n,2)-\log 2)$, and, furthermore, $x^2 + y^2$ cannot exceed $2\exp(S(n;4,1))$. Thus, from these observations,

$$x^2 - y^2 \geq \exp(S(n;4,3) + S(n,2) - \log 2)$$

and

$$x^2 + y^2 \leq 2\exp(S(n;4,1)) .$$

Therefore, from (4.12),

$$\begin{aligned} \log n! &\leq \log(x^2-y^2) + \log 2 + S(n;4,1) \\ &\leq 2\log 2 + 2S(n;4,1) . \end{aligned} \tag{4.13}$$

Inequality (4.13) implies

$$\log n! \leq 2\log 2 + 2n \sum_{\substack{p\equiv 1(4) \\ p^r \leq n \\ r \geq 1}} \frac{\log p}{p^r} , \tag{4.14}$$

and, from a well-known estimate for $\log n!$ [18; p. 257], together with inequality (4.1), we have

$$n\log n - n + \frac{1}{2}\log n + \log\sqrt{2\pi} < n\log n - 1.19182n + \frac{1.82n}{\log n} + 8\sqrt{n} + 2\log 2$$

for $n \geq 961$. Hence

$$.19182n < \frac{1.82n}{\log n} + 8\sqrt{n} - \frac{1}{2}\log n + .47$$

for $n \geq 961$. Since this inequality is false for $n \geq e^{12}$ it follows that $n! = x^4 - y^4$ has no solutions in positive integers n,x,y for $n \geq e^{12}$.

For the purpose of checking inequality (4.14) for $n < e^{12}$ we use the following results of J. Rosser and L. Schoenfeld [22] :

$$\psi(x) \geq \theta(x) > x - 2.05282\sqrt{x} \qquad \text{for } x \leq 10^8 ,$$

$$\sum_{p \leq x} \frac{\log p}{p} < \log x + E + \frac{2}{\sqrt{x}} \qquad \text{for } 114 \leq x \leq 10^8 ,$$

where $E \leq -1.33258$.

Repeating the procedure used in obtaining (4.11) with the above inequalities in place of (4.7) and (4.9) we have

$$\sum_{\substack{p \equiv 1(4) \\ p^r \leq n \\ r \geq 1}} \frac{\log p}{p^r} < \frac{1}{2}\log n - .59571 + \frac{3.554}{\sqrt{n}} + \frac{.637\log n + 17}{n} + \frac{238}{n^2}$$

for $961 \leq n \leq 10^8$. However, when this inequality is used in (4.14) the resulting inequality

$$.19182n < 7.108\sqrt{n} + (1.274)\log n + 35 + \frac{576}{n}$$

is false for $n \geq 2000$.

The inequality

$$\log n - 1 - \frac{\log 4}{n} < \frac{\log n! - \log 4}{n} \leq 2 \sum_{\substack{p \equiv 1(4) \\ p^r \leq n \\ r \geq 1}} \frac{\log p}{p^r}$$

was checked directly for $n < 2000$ and found to be false for $2 < n < 2000$. Hence (4.14) does not hold for $2 < n < 2000$.

Collecting results, we find that $n! = x^4 - y^4$ cannot hold for $n > 2$, and since $1 \nmid x^4 - y^4$, $2 \nmid x^4 - y^4$ for positive integers x and y it follows that $n! = x^4 - y^4$ has no solutions in positive integers n, x, y with $(x,y) = 1$.

APPENDIX

Values of $\psi(x;4,1)$ and $\psi(x;4,3)$ for $x \leq 600$.*

x	$\psi(x;4,1)$	x	$\psi(x;4,3)$
5	1.60944	3	1.09861
9	2.70805	7	3.04452
13	5.27300	11	5.44242
17	8.10622	19	8.38686
25	9.71566	23	11.52236
29	13.08296	27	12.62097
37	16.69388	31	16.05496
41	20.40746	43	19.81617
49	22.35337	47	23.66632
53	26.32367	59	27.74386
61	30.43455	67	31.94856
73	34.72501	71	36.21124
81	35.82362	79	40.58069
89	40.31226	83	44.99954
97	44.88698		

* This table was compiled with the aid of the factorization table included in [18].

x	$\psi(x;4,1)$
101	49.50211
109	54.19347
113	58.92087
121	61.31877
125	62.92821
137	67.84820
149	72.85216
157	77.90842
169	80.47337
173	85.62667
181	90.82518
193	96.08788
197	101.37109
229	106.80482
233	112.25587
241	117.74068
257	123.28977
269	128.88449
277	134.50852
281	140.14688
289	142.98010
293	148.66028

x	$\psi(x;4,3)$
103	49.63428
107	54.30712
127	59.15132
131	64.02653
139	68.96101
151	73.97830
163	79.07206
167	84.19006
179	89.37746
191	94.62974
199	99.92305
211	105.27492
223	110.68210
227	116.10706
239	121.58353
243	122.68214
251	128.20760
263	133.77976
271	139.38189
283	145.02735

x	$\psi(x;4,1)$	x	$\psi(x;4,3)$
313	154.40649	307	150.75421
317	160.16540	311	156.49401
337	165.98549	331	162.29614
349	171.84057	343	164.24205
353	177.70705	347	170.09138
361	180.65149	359	175.97471
373	186.57308	367	181.88008
389	192.53667	379	187.81763
397	198.52062	383	193.76567
401	204.51459	419	199.80355
409	210.52831	431	205.86967
421	216.57095	439	211.95418
433	222.64170	443	218.04776
449	228.74873	463	224.18550
457	234.87342	467	230.33184
461	241.00683	479	236.50355
		487	242.69182
		491	248.88827
		499	255.10089

x	$\psi(x;4,1)$	x	$\psi(x;4,3)$
509	247.23929	503	261.32149
521	253.49246	523	267.58108
529	256.62796	547	273.88554
541	262.92139	563	280.21883
557	269.24397	571	286.56623
569	275.58786	587	292.94126
577	281.94571	599	299.33653
593	288.33091		

BIBLIOGRAPHY

[1] K. Appel and J. Rosser, "Table for estimating functions of primes", Inst. for Defence Analyses (Comm. Research Div., Von Neumann Hall, Princeton, New Jersey) Tech. Rep. No. 4 (1961).

[2] P. Bateman and R. Horn, "A Heuristic Asymptotic Formula Concerning the Distribution of Prime Numbers", Math. Comp., Vol. 16 (1962), 363-367.

[3] R. Breusch, "Zur Verallgemeinerung des Bertrandschen Postulates, dass zwischen x und $2x$ stets Primzahlen liegen", Math. Zeitschrift, Vol. 34 (1932), 505-526.

[4] R. Breusch, "An Asymptotic Formula for Primes of the Form $4n+1$ ", Michigan Mathematical Journal, Vol. 11 (1964), 311-315.

[5] H. Brocard, Nouv. Corresp. Math., Vol. 2 (1876), 287.

[6] H. Brocard, Mathesis, Vol. 7 (1887), 280.

[7] P. Erdös, "A theorem of Sylvester and Schur", Journal London Math. Soc., Vol. 9 (1934), 282-288.

[8] P. Erdös. "Über die Primzahlen gewisser arithmetischer Reihen", Math. Zeitschrift, Vol. 39 (1935), 473-491.

[9] P. Erdös, "On Consecutive Integers", Nieuw Archief voor Wiskunde, (3) III (1955), 124-128.

[10] P. Erdös and R. Obláth, "Über diophantische Gleichungen der Form $n! = x^p \pm y^p$ und $n! \pm m! = x^p$", Acta Scientiarum Mathematicarum (Szeged.), Vol. 8 (1937), 241-255.

[11] M. Faulkner, "Some Numerical Studies in the Theory of Numbers", Master's Thesis, University of Alberta, (1963).

[12] M. Faulkner, "On a Theorem of Sylvester and Schur", Journal London Math. Soc., Vol. 41 (1966), 107-110.

[13] H. Gupta, "On the Diophantine Equation $m^2 = n! + 1$", American Math. Monthly, Vol. 43 (1936), 32-34.

[14] G. Hardy and E. Wright, "An Introduction to the Theory of Numbers", Oxford University Press (1960), 241-243, 270-271.

[15] M. Kraitchik, "Recherches sur la Theorie des Nombres", Vol. 1, Gauthier-Villars, Paris, (1924), 38-41.

[16] D. Lehmer, "Some high-speed logic", Proc. Symp. in Applied Math., Vol. 15 (1963), 141-145.

[17] K. Molsen, "Zur Verallgemeinerung des Bertrandschen Postulates", Deutsche Mathematik, Vol. 6 (1941), 248-256.

[18] P. Morse et. al., "Handbook of Mathematical Functions", National Bureau of Standards Applied Mathematics Series 55, (1964).

[19] L. Moser, "Insolvability of $\binom{2n}{n} = \binom{2a}{a}\binom{2b}{b}$ ", Canadian Math. Bull., Vol. 6 (1963), 167-169.

[20] G. Ricci, "Sul teorema di Dirichlet relativo alla progressione aritmetica", Boll. Un. Mat. Ital., Vol. 12 (1933), 304-309.

[21] G. Ricci, "Sui teoremi di Dirichlet e di Bertrand-Tchebychef relativi alla progressione aritmetica", Boll. Un. Mat. Ital., Vol. 13 (1934), 7-17.

[22] J. Rosser and L. Schoenfeld, "Approximate formulas for some functions of prime numbers", Illinois Journal of Math., Vol. 6 (1962), 64-94.

[23] D. Roux, "Sui numeri primi delle progressioni aritmetiche", Boll. Un. Mat. Ital., (3) Vol. 11 (1956), 55-63.

[24] A. Schinzel and W. Sierpiński, "Sur certaines hypothèses concernant les nombres premiers", Acta Arithmetica, Vol. 4 (1958), 185-208.

[25] A. Schinzel, "Remarks on the paper "Sur certaines hypothèses concernant les nombres premiers", Acta Arithmetica, Vol. 7 (1961), 1-8.

[26] I. Schur, "Einige Sätze über Primzahlen mit Anwendungen auf Irreduzibilitätsfragen. I." Sitzungsberichte der preussischen Akademie der Wissenschaften, Phys. Math. Klasse, Vol. 14 (1929) 125-136.

[27] J. Sylvester, "On Tchebycheff's Theory of the Totality of the Prime Numbers Comprised within Given Limits", American Journal of Math., Vol. 4 (1881), 230-247.

[28] J. Sylvester, "On arithmetical series", Messenger of Math., Vol. 21 (1892), 1-19, 87-120.

[29] P. Tschebyschef, "Mémoire sur les nombres premiers", Liouv. Journ., Vol. 17 (1852).

[30] W. Utz, "A conjecture of Erdös concerning consecutive integers", American Math. Monthly, Vol. 68 (1961), 896-897.

[31] E. Waage, "Zur Tschebyschef'schen Primzahlenthorie", Sitzungsberichte, Akademie der Wissenschaften in Wien, Mathematische-Naturwissenschaftlichen Klasse, Vol. 122, Abt. II a (1913), 701-719.

[32] E. Waage, "Zur Tschebyschef'schen Primzahlentheorie II", Sitzungsberichte, Akademie der Wissenschaften in Wien, Mathematische-Naturwissenschaftlichen Klasse, Vol. 123, Abt. II a (1914), 493-510.